CUT POINTS IN GENERAL TOPOLOGICAL SPACES

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1. *Introduction.*—The identification of those connected and locally connected spaces in which the intersection of every pair (or collection) of connected sets is connected (possibly empty) is of importance in the study and use of unicoherence. Solving this problem involves a restudy and extension to general spaces of many of the results about cut points, conjugacy of point pairs, structure of locally connected spaces relative to its cut points, and cyclic elements which are well known[2] for separable metric spaces. This will be accomplished in the present paper. In a general way it may be said that results not involving countability extend to all connected and locally connected Hausdorff spaces, indeed to spaces satisfying a greatly weakened Hausdorff-type axiom (see §3 below). For spaces not satisfying this axiom, little if anything of significance can be proved without restricting the space in some other way (e.g., see §2).

Let M be any connected T_1 space, and for each pair of distinct points $a,b \in M$ let $E(a,b)$ be the set of all points in M, each of which separates a and b in M. For each $x \in E(a,b)$ we choose a definite separation

$$M - x = M_a(x) + M_b(x)$$

of $M - x$ between a and b, $a \in M_a(x)$, $b \in M_b(x)$. The set $E(a,b)$ is ordered by the definition $x < y$ if and only if $x \in M_a(y)$. Equivalently, $x < y$, provided that $y \in M_b(x)$. Also if we set $a < x$ and $x < b$ and $a < b$ for all $x \in E(a,b)$, we have an ordering of $E(a,b) + a + b$. This ordering is *natural* in the sense that for any x in the set, the set of all its predecessors is open, as is also the set of all its successors.

Throughout this paper all spaces are assumed to be T_1 spaces. The boundary of an open set U is designated by $Fr(U)$, and a connected open set is called a *region.*

2. *Existence of Non-Cut Points.*—As is well known, a connected space may consist entirely of cut points or, likewise, entirely of non-cut points. However, every nondegenerate continuum (= compact connected metric space) must have at least two non-cut points.[1] Using the maximality form of the axiom of choice, we include next a proof of this latter conclusion for arbitrary compact connected nondegenerate T_1 spaces.

Every nondegenerate compact connected T_1 space M has at least two non-cut points.

Proof: Suppose not. Take $a \in M$ so that a is a non-cut point of M if M has one such point. For each $x \in M - a$ we get a separation $M - x = A(x) + B(x)$, where $A(x) \supset a$. Define $C(x) = A(x) + x$ for each $x \in M - a$. We note that each $C(x)$ is a connected closed *proper* subset of M. Let the sets $\{C(x)\}$ be partially ordered by inclusion. Let $\mathcal{W}$ be a maximal simply ordered chain of sets in $\{C(x)\}$.

Then $W = \bigcup C, C \epsilon \mathcal{W}$, is connected, since each $C(x)$ contains a. We assert that $W = M$. For if there exists $x \epsilon M - W$, we have $W \subset A(x)$; and if $y \epsilon B(x)$, $C(y) = A(y) + y \supset A(x) + x \supset W$. This is impossible since $\mathcal{W}$ is a maximal chain, whereas $\mathcal{W} + \{C(y)\}$ would be a chain properly containing $\mathcal{W}$. Thus $W = M$. Further, each $p \epsilon M$ is interior to some $C(x)$ in $\mathcal{W}$. For take $p \epsilon C(x) \epsilon \mathcal{W}$. If $p \neq x$, this is true because $A(x)$ is open; and for $p = x$ we need only take a $C(y)$ in $\mathcal{W}$ which meets $B(x)$ and we have $x \epsilon C(y)$ and $x \neq y$ so that $x \epsilon A(y)$.

However, no finite subcollection of $\mathcal{W}$ can cover M, because if so, the largest element in this subcollection would be identical with W and thus with M, whereas each set in $\mathcal{W}$ is a *proper* subset of M. This contradicts the compactness of M.

3. *Conjugacy—Axiom H_1.*—Let M be a connected and locally connected T_1 space. Two points $a,b \epsilon M$ are *conjugate* provided that no point of M separates them in M, i.e., $E(a,b) = \Phi$. Also a and b are *separable* or *inseparable* according as there do or do not exist two disjoint open sets about a and b, respectively.

AXIOM H_1. *For any two distinct inseparable points in the boundary of a region R there exists a point of R which is conjugate to both of these points.*

Remark: Since there are no inseparable point pairs in a Hausdorff space, *every Hausdorff space satisfies axiom H_1.* However, there are many H_1 spaces which are not Hausdorff spaces. For example, any infinite set in which the empty set and sets with finite complements are defined as open sets is such a space. This space is also connected and locally connected.

A T_1 space satisfying Axiom H_1 will be called an *H_1 space.*

THEOREM H_1. *For any two distinct conjugate points in the boundary of a region R in a connected, locally connected, H_1 space M three exists a point of R conjugate to each of these points.*

Proof: Let a and b be two such points. By Axiom H_1 we may assume a and b separable so that there exist disjoint regions R_a and R_b about a and b, respectively. Take $q \epsilon R \cdot R_a$ and let Q be the component of $R - R \cdot \bar{R}_b$ containing q. Then R contains a boundary point p of Q and $p \epsilon \bar{R}_b$. Then p is conjugate to both a and b. For suppose there is a separation $M - x = M_p + M_0$ where $p \epsilon M_p$ and $(a + b) \cdot M_0 \neq \Phi$. Since a and b are conjugate, we must have $a + b \subset M_0 + x$. Thus x lies in each of the connected sets $R_b + p$ and $R_a + Q + p$. This is impossible as these sets meet only in p and $p \neq x$.

4. *Closedness and Compactness of $E(a,b) + a + b$.*—These properties of this set are well known in case M is connected, locally connected separable, and metric.[2] However, in non-Hausdorff spaces they may fail to hold.

EXAMPLE. *There exists a connected and locally connected perfectly separable T_1 space containing a pair of points a and b for which $E(a,b) + a + b$ is neither closed nor compact.*

Let M consist of the graph G of the function $y = \sin 1/x$ together with the two points $p = (0,1)$, $q = (0,-1)$. Let a basis for the open sets in M consist of all ordinary open connected segments on G between points with rational abscissas of the same sign together with the sets obtained by adjoining p (resp. q) to the part of G between $X = -r$ and $x = r$ for each positive rational r. It is readily verified that M has all our required properties. If a and b are taken on G on opposite sides of $x = 0$, neither p nor q separates a and b so that $E(a,b) + a + b$ is not closed; and it is equally apparent that it is also noncompact.

THEOREM 4.1. *In a connected and locally connected H_1 space M, every set $K = E(a,b) + a + b$ is closed.*

Proof: Suppose there is a point $p \epsilon \bar{K} - K$. For each $x \epsilon E(a,b)$ take a definite separation $M - x = M_a(x) + M_b(x)$ between a and b. Define E_a to be the set of all $x \epsilon E(a,b)$ such that $p \epsilon M_b(x)$ and E_b as all $x \epsilon E(a,b)$ with $p \epsilon M_a(x)$. Then p is a limit point of at least one of the sets E_a and E_b. The two cases are alike, so we suppose $p \epsilon \bar{E}_a$. Then if $R = \bigcup M_a(x), x \epsilon E_a$, R is open. It is also connected because for each $x \epsilon E_a$ there is a $y \epsilon E_a$ with $y \epsilon M_b(x)$ so that the connected set $M_a(x) + x$ lies in $M_a(y)$ and thus in R.

Since p is not in K and thus does not separate a and b, the boundary $Fr(R)$ of R contains at least one point q in addition to p. Now let r be any point whatever of R. Then $r \epsilon M_a(x)$ for some $x \epsilon E_a$. Since p and q both lie in $M_b(x)$ and $M_b(x) + x$ is connected and does not contain r, it follows (i) that x separates r from both p and q, and (ii) that r does not separate p and q in M. Since any point separating p from q would necessarily lie in R, it follows by (ii) that p and q are conjugated. However, by (i) no point whatever of R can be conjugate to both p and q. This contradicts Theorem H_1 of §3.

THEOREM 4.2. *In a connected and locally connected T_1 space M, every set $K = E(a,b) + a + b$ which is closed is compact.*

Proof: Obviously we may suppose $E(a,b) \neq \Phi$. Let $\mathcal{G}$ be any open covering of K. For each $x \epsilon K$ we denote the "segment" $E(a,x) + a + x$ of $E(a,b) + a + b$ by ax. Let K_a be the set of all $x \epsilon K$ such that ax is covered by some finite subcollection of $\mathcal{G}$. We will show that $K - K_a = \Phi$. Suppose not.

Then K_a has a last point p. In case $K_a = a$, obviously a is this last point. Otherwise, the union U_a of all sets $M_a(x)$ for $x \epsilon K_a - a$ is a nonempty, open proper subset of M. Thus some $p \epsilon M - U_a$ is a limit point of U_a. Now if p were not in K, a region R_p about p not meeting the closed set K would meet some $M_a(x)$ in U_a and thus would lie wholly in $M_a(x)$ so that p would be in U_a. Thus $p \epsilon K$. Let G be a set of $\mathcal{G}$ containing p and let Q be a region containing p and lying in G. Then Q meets some set $M_a(x)$ in U_a and thus contains x because either $p = x$ or x separates $M_a(x)$ and p in M. Now ax is covered by a finite subcollection G_1 of $\mathcal{G}$; and if G is adjoined to G_1, we get a finite subcollection of $\mathcal{G}$ covering ap because any point of $ap - ax - p$ separates x and p in M and thus lies in G. Hence $p \epsilon K_a$; and since any other point x of $K_a - a$ lies in the connected set $M_a(x) + x$ which is contained in $M_a(p)$, p must be the last point of K_a.

It follows similarly that $K - K_a$ has a first point q. This is obvious if $b = K - K_a$. Otherwise the union U_b of all sets $M_b(x)$ for $x \epsilon K - K_a - b$ is a nonempty open proper subset of M and the existence of q follows by essentially the same argument as just given for that of p.

However, this situation involves a contradiction. For clearly a set of $\mathcal{G}$ containing q adjoined to a finite subcollection of covering $ap = K_a$ gives a finite collection in $\mathcal{G}$ covering aq contrary to $q \epsilon K - K_a$.

Combining Theorems (4.1) and (4.2), we get

THEOREM 4.3. *In a connected and locally connected H_1 space, every set $E(a,b) + a + b$ is closed and compact.*

5. *A-Sets.*—A closed set A in a connected T_1 space X is an *A-set* provided that $X - A$ is the union of a collection of open sets each bounded by a single point of A.

REMARK. *If an A-set A meets each of two interesecting connected sets M and N, then it meets their intersection.*

For let $x \epsilon M{\cdot}N$. If x is not in A, it is in an open set Q in $X - A$ whose boundary is a single point p of A. Both M and N must contain p since each meets both x and A.

THEOREM 5.1 (FINITE INTERSECTION THEOREM). *The intersection of any two A-sets is either empty or an A-set.*

Proof: Let A and B be A-sets in X where X is connected and $A{\cdot}B \neq \Phi$. Denote the open sets with single point boundaries covering $X - A$ and $X - B$, respectively, by $\{Q\}$ and $\{R\}$.

Now for any x not in $A \cdot B$ either $x \epsilon X - A$ or $x \epsilon X - B$, say $x \epsilon X - A$. Let Q_x be a set in $\{Q\}$ containing x with boundary point q in A. If $q \epsilon B$, we set $Q_x = P$. If not, by the Remark, B does not meet the connected set $Q_x + q$ and thus some set R_q of $\{R\}$ contains this set. Let r be the boundary point of R_q in B. Since A meets both $R_q + r$ and $X - R_q$, by the Remark it must contain r. Thus if in this case we set $R_q = P$, then in either case P is an open set in $X - A{\cdot}B$ containing x and whose boundary is a single point of $A{\cdot}B$. Thus $A \cdot B$ is an A-set.

REMARK. *In a connected and locally connected T_1 space M a nonempty closed set A is an A-set if and only if each component of $M - A$ has exactly one boundary point.*

THEOREM 5.2 (INTERSECTION THEOREM). *In a connected and locally connected H_1 space M, any intersection of A-sets is either empty or an A-set.*

Proof: Let $A = \bigcap A_\lambda$, where each A_λ is an A-set and $A \neq \Phi$. Take a component Q of $X - A$. Suppose contrary to our theorem that Q has two distinct boundary points p and q in A. Let r be any point whatever of Q. For some λ we have $r \epsilon X - A_\lambda$. Let Q_r be the component of $X - A$ containing r and let x be its boundary point in A_λ. Then since $p + q$ lies in the connected set $M - Q_r$ whereas $r \epsilon Q_r$, we have that r cannot separate p and q in M and that r is not conjugate to both p and q. Since any point separating p and q would lie in Q, it follows that p and q are conjugate and that no point of Q is conjugate to both of them. This contradicts Theorem H_1 of §3.

Note: The general intersection theorem does not hold in the absence of the local connectedness condition on the space. For let X be the graph of the function $y = \sin 1/x$ together with the closed interval A of the y-axis from -1 to 1 with the usual topology of the plane. Then for each $a > 0$ the part of X with abscissas in the closed interval from $-a$ to a is an A-set, whereas the intersection A of all these sets clearly is not an A-set.

Likewise the theorem fails to hold if the H_1 condition is omitted. For if M is the space described in the example of §4, again for each $a > 0$ the part of M

between $-a$ and a is an A-set but the intersection of all these sets is not an A-set.

THEOREM 5.3. *If A is an A-set in a connected and locally connected T_1 space M, then for any connected set Z whatever in M, $A \cdot Z$ is connected (possibly empty).*

For suppose we have a separation

$$A \cdot Z = A_1 + A_2.$$

Then Let $Z_1 = A_1 + Q_1 \cdot Z$, $Z_2 = A_2 + Q_2 \cdot Z$ where Q_1 and Q_2 are the unions, respectively, of all components of $M - A$ with boundary point in A_1 and A_2. It is readily verified that $Z = Z_1 + Z_2$ and that this is a separation, contrary to connectedness of Z.

COROLLARY. *In a connected and locally connected T_1 space M, every A-set is connected and locally connected.*

Remark: This theorem and corollary do not hold without local connectedness on M. For let the space X consist of the union of two sequences of intervals in 3-space, one of these consisting of intervals on the xy plane joining the origin a to the points $(1,1/n,0)$, $n = 1, 2, \ldots$, and the other of intervals in the xz plane joining the point $b = (1,0,0]$ to the point $(0,0,1/n)$, $n = 1, 2, \ldots$. The set consisting of the points a and b alone is an A-set in this connected space.

6. *Chains $C(a,b)$—E_0-Sets.*—Throughout this section N will denote a connected and locally connected H_1 space. If a and b are distinct conjugate points in N, $C(a,b)$ will denote the set of all points of N which are conjugate to both a and b. (In general, $C(a,b)$ will denote the intersection of all A-sets in N containing $a + b$. It will be shown in §7 that this set is identical with $C(a,b)$ as just defined in case a and b are conjugate.) A connected nondegenerate subset of N which has no cut point and is maximal in M relative to these properties will be called an E_0-set.

THEOREM 6.1. *Every set $C(a,b)$ is an A-set.*

Proof: If $z \in N - C(a,b)$, z fails to be conjugate to at least one of the points a and b, say to a. Thus there is a separation $N - x = N_z + N_a$ between z and a for some $x \in N$. Since obviously $N_z \cdot C(a,b) = \Phi$ and N_z is open, it follows that $C(a,b)$ is closed.

Now let R be any component of $N - C(a,b)$. Suppose, contrary to our conclusion, that R has two distinct boundary points p and q in $C(a,b)$. Then let r be any point whatever of R. There exists a separation

$$N - x = N_r + N_0$$

for some $x \in N$ where $r \in N_r$ and $(a + b) \cdot N_0 \neq \Phi$, because r is not in $C(a,b)$. Then since $p,q \in C(a,b)$ we must have

$$p + q + a + b \subset C(a,b) \subset N_0 + x.$$

Accordingly r does not separate p and q and r is not conjugate to both p and q. Since r is an arbitrary point of R, p and q must be conjugate and yet no point of R is conjugate to them both. This is contrary to Theorem H_1 of §3.

COROLLARY. *Every set $C(a,b)$ in N is connected and locally connected.*

THEOREM 6.2. *Every set $C(a,b)$ in N (a and b distinct conjugate points) is an*

E_0-set. Conversely, every E_0-set E is identical with $C(a,b)$ for any two distinct points $a,b \in E$.

Proof: To show that $C(a,b)$ has no cut point, let $x \in C(a,b)$. Then $C(a,b) - x$ must be in some single component Q of $N - x$ since each of its points is conjugate to both a and b. Thus

$$C(a,b) - x = Q \cdot C(a,b),$$

and the set on the right is connected by Theorems 6.1 and 5.3. Since any connected set in N properly containing $C(a,b)$ meets some component of $N - C(a,b)$ and thus has a cut point, it follows that $C(a,b)$ is an E_0-set.

Now let E be any E_0-set and take any two distinct points $a,b \in E$. Then clearly $E \subset C(a,b)$ since any two points in E are conjugate. Thus since by the above $C(a,b)$ is an E_0-set, we have $E = C(a,b)$.

COROLLARY. *If a and b are distinct conjugate points of N, $C(a,b)$ is uniquely determined by any pair of its distinct points. Any two such sets $C(a,b)$ (or E_0-sets) have at most one common point.*

Note: The E_0-sets, now identified with the sets $C(a,b)$ for a and b conjugate points, will be referred to as the *true cyclic elements* of N.

7. *Cyclic Chains.*—Again in this section N will denote a connected and locally connected H_1 space. For any two points a and b of N, the *cyclic chain* $C(a,b)$ is the intersection of all A-sets in N containing $a + b$. Since by §5, $C(a,b)$ is an A-set, it may be called the *least* A-set containing $a + b$. We note also that in case a and b are *distinct* and conjugate, $C(a,b)$ is $a + b$ identical with the set $C(a,b)$ as defined in §6, since every A-set containing $a + b$ must contain all of $C(a,b)$ as defined in §6 because $a + b$ has no cut point by Theorem 6.2.

THEOREM 7.1. *For any two points $a,b \in N$, we have*

$$C(a,b) = E(a,b) + a + b + C,$$

where C is the union of all true cyclic elements (= E_0-sets) of N each meeting $E(a,b) + a + b$ in exactly two points.

Proof: Let A and B denote the sets on the left and right, respectively, of this relation. To show $A = B$ we prove that (*i*) B is an A-set (so that $B \supset A$), and (*ii*) $A \supset B$.

To prove (*i*) we first show that B is closed. Suppose that there exists $p \in \bar{B} - B$. Since $E(a,b) + a + b$ is closed, there is a region R about p not meeting $E(a,b) + a + b$. Then R meets some pair E_1,E_2 of distinct E_0-sets in B. However, since $E_1 \cdot E_2$ is at most one point, $E_1 + E_2$ contains at least three distinct points x, y, z of $E(a,b) + a + b$ and we may suppose these ordered so that y separates x and z in N. Since y is not in R, this clearly is impossible because $E_1 + E_2 - y + R$ is connected. Thus B is closed. It remains to show that if Q is any component of $N - B$, $Fr(Q)$ is a single point. If, on the contrary, $Fr(Q)$ contained two distinct points p and q, p and q cannot be conjugate because Q would be a component of the complement of the E_0-set determined by p and q. Thus some point x separates p and q. However, x would then separate a and b and be in $E(a,b)$ since otherwise it would separate either p or q from $a + b$, which is clearly impossible. Thus we have a contradiction. This proves (*i*).

To prove (*ii*), note first that $A \supset E(a,b) + a + b$ since A is connected. Also if E is an E_0-set in B, it must be in A because it meets A in at least two distinct points and A is an A-set. Otherwise the boundary point of any component of $N - a$ meeting E would separate E.

COROLLARY 7.2. *If for two points, $a,b \in N$, no two points of the set $K = E(a,b) + a + b$ are conjugate, then $C(a,b) = K$. Thus K is a compact connected and locally connected Hausdorff space such that every point of $K - a - b$ separates a and b in K.*

Thus K is a "simple arc" from a to b in the usual sense except that it may not be separable. Indeed K will be homeomorphic with the unit interval if and only if it is separable.

8. *An Example.*—In order to illustrate the great generality of the spaces to which the structure theory developed above is applicable, and at the same time to show that the usual countability results do not hold in the absence of a separability assumption, we describe next an example of interest.

The space X will consist of the points of a torus T in E^3 and we make use of a toral coordinate system (u,v) on T, $0 \leq u,v < 2\pi$ as well as of the Euclidean distance or vector length $|x - y|$ in E^3. For any two points $x = (u,v)$ and $y = (u',v')$ on T let x_0 and y_0 be the points $(u,0)$ and $(u',0)$, respectively, of T. We then define a metric $\rho(x,y)$ in X by the equations:

$$\rho(x,y) = |x - y|, \quad \text{if } u = u' \text{ or if } v = v' = 0;$$

and

$$\rho(x,y) = |x - x_0| + |x_0 - y_0| + |y_0 - y|, \quad \text{otherwise.}$$

It is readily verified that with this distance function X is a connected and locally connected metric space. Also X contains isometrically the circle $C{:}v = 0$ and each of the circles $C_\lambda{:}u = \lambda$ for $0 \leq \lambda < 2\pi$. Further,

(1) Each of the circles C and C_λ, $0 \leq \lambda < 2\pi$, is a true cyclic element of X.

(2) Each point of C is a cut point of X, cutting it into exactly two components.

(3) No proper subarc of C is an A-set in X even though it is a continuum of cut points of X.

(4) For each λ, $0 \leq \lambda < 2\pi$, $C_\lambda - (\lambda,0)$ is an open set.

(5) Every point of X is a local cut point of X.

9. *The Connected Intersection Property.*—We can now give a quick answer to the identification question raised in the introduction, at least insofar as it applies to Hausdorff spaces.

THEOREM 9.1. *In order that a connected and locally connected topological space M be a Hausdorff space in which the intersection of every pair (indeed every collection) of connected sets is connected, it is necessary and sufficient that no two distinct points of M be conjugate.*

If no two points of M are conjugate, then for $a,b \in M$ some $x \in M$ gives a separation $M - x = M_a + M_b$ between a and b with M_a and M_b open. Thus M is a Hausdorff space. Also for any connected set N in M and any pair of points $a,b \in N$ we have $K = C(a,b) = E(a,b) + a + b \subset N;$ and by (7.2), K is connected and is a type of "simple arc" as there described. Thus if $\mathfrak{N} = \{N_\lambda\}$ is any collection of connected sets in M, for any two points $a,b \in N = \bigcap N_\lambda$ we have $K = C(a,b) \subset N$ for every λ so that $K \subset N$. Thus N is not only connected but is "arcwise connected" in the above sense.

On the other hand, suppose now that M is a Hausdorff space containing a pair of distinct conjugate points a and b, where M is also connected and locally connected. Then by §6, $C(a,b) = E$ is a nondegenerate connected and locally connected set having no cut point. Let R be a region in E such that there exists a point x in $E - \bar{R}$. Then if Q is the component of $E - \bar{R}$ containing x, Q must have at least two distinct boundary points p and q in $\bar{R}$, since otherwise E would have a cut point. Then $\bar{R}$ and $Q + p + q$ are connected sets in M meeting in the disconnected set $p + q$. Thus we have shown that if each pair of connected sets in M has a connected intersection, then no two distinct points of M can be conjugate.

COROLLARY 9.2. *If the intersection of every pair of connected sets in a connected and locally connected Hausdorff space is connected, the same is true of the intersection of an arbitrary collection of connected sets in this space.*

Concerning intersections of arbitrary collections of connected sets, we have at once that in a connected and locally connected T_1 space M, if the intersection of every collection of connected sets in M is connected, M has no pair of distinct conjugate points and M is a Hausdorff space.

To prove this, suppose a and b are distinct conjugate points in M. Then for each $x \in M - a - b$, a and b lie together in a single component Q_x of $M - x$. Since clearly $Q = \bigcap Q_x = a + b$, Q fails to be connected.

Combining this conclusion with Corollary 9.2, we get

THEOREM 9.3. *In order that the intersection of an arbitrary collection of connected sets in a connected and locally connected T_1 space be connected, it is necessary and sufficient that $E(x,y) \neq \Phi$ for every pair $x,y \in M$.*

Note: Of course, either property implies that M is a Hausdorff space.

Also we note that the results of this section do not hold without the local connectedness assumption. For *there exists a connected subset M of the plane with $E(x,y) \neq \Phi$ for all $x,y \in M$ but with $E(a,b) + a + b$ not connected for a pair a,b in M. Also M contains two connected subsets whose intersection is not connected.* To see this, let K be the union of the closed interval from $a = (0,0)$ to $(1,0)$ and the open interval from $(1,1)$ to $(2,1) = b$ together with the end point b. Then let T_n be the closed interval from $(1 + 1/n,0)$ to $(1 + 1/n,1)$, $n = 1,2, \ldots$ and let $M = K + \bigcup T_n$. Clearly M meets all our conditions and $E(a,b) + a + b = K$ is not connected. Also the connected subsets $M_1 = K + \bigcup T_n$, n odd, and $M_2 = K + \bigcup T_n$, n even, meet in exactly the disconnected set K.

[1] Moore, R. L., "Concerning simple continuous curves," *Trans. Am. Math. Soc.*, **21**, 333–347 (1920).

[2] Whyburn, G. T., "On the structure of connected and connected im kleinen point sets," *Trans. Am. Math. Soc.*, **32**, 926–943 (1930).